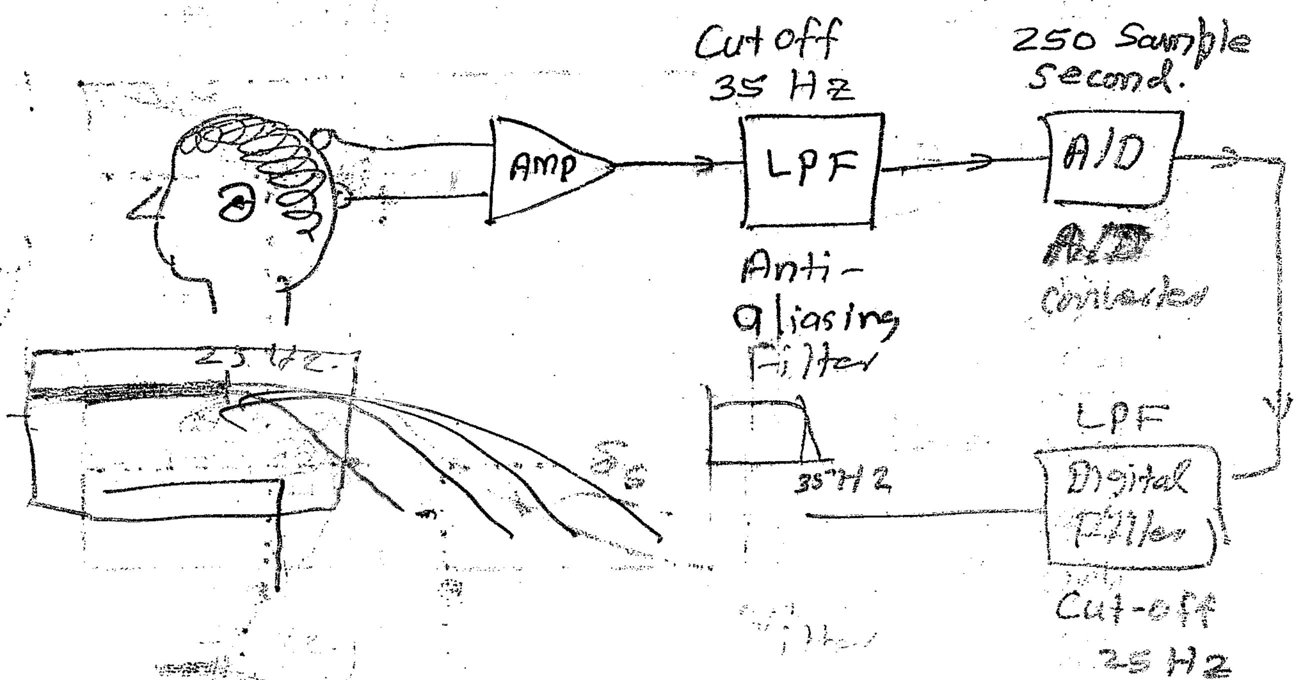
Example 8.14 Digital Butterworth Filter

* EEG Signals ~ 0 – 25 Hz



* Design a Digital Butterworth Filter with cutoff 25 Hz.

* General Analogue prototype (Continuous Time)

Unity Gain CT Butterworth filter of order N

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

1). In order to design the digital filter starting from an analog (CT) prototype, we need to know N and Ω_c.

2) The digital filter has to be specified in the digital domain. That is, filter

- <u>In the given problem:</u>

  f_s = Sampling Frequency = 250 Hz

- <u>Desired ω_p</u>

  $$\omega_p = \Omega_p \cdot T = (2\pi \times 25)\frac{1}{250}$$

  $$= 0.2\pi$$

- <u>Pick a value for ω_s</u>

  Let $\omega_s = 0.3\pi$

- <u>Choose δ_p and δ_s</u>

  Let $\delta_p = -1$ dB

  δ_s = [illegible] dB

- Now the digital specification are complete.

- Translate the specs. to analog domain.

  $$\omega_p \longrightarrow \Omega_p = \frac{\omega_p}{T} = \frac{0.20\pi}{(1/250)} = 157 \text{ rad/s}$$

  $$\omega_s \longrightarrow \Omega_s = \frac{\omega_s}{T} = \frac{0.3\pi}{(1/250)} = 235.6 \text{ rad/s}$$

- When $\Omega = \Omega_p$, gain $= -1$ dB

$$20 \log_{10} |H(\Omega_p)| = -1$$

$$10 \log_{10} |H(\Omega_p)|^2 = 10 \log \left(\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \right)$$

$$\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = 10^{-0.1} \quad \text{——} \quad (1)$$

- When $\Omega = \Omega_s$, gain $= -15$ dB

$$\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = 10^{-1.5} \quad \text{——} \quad (2)$$

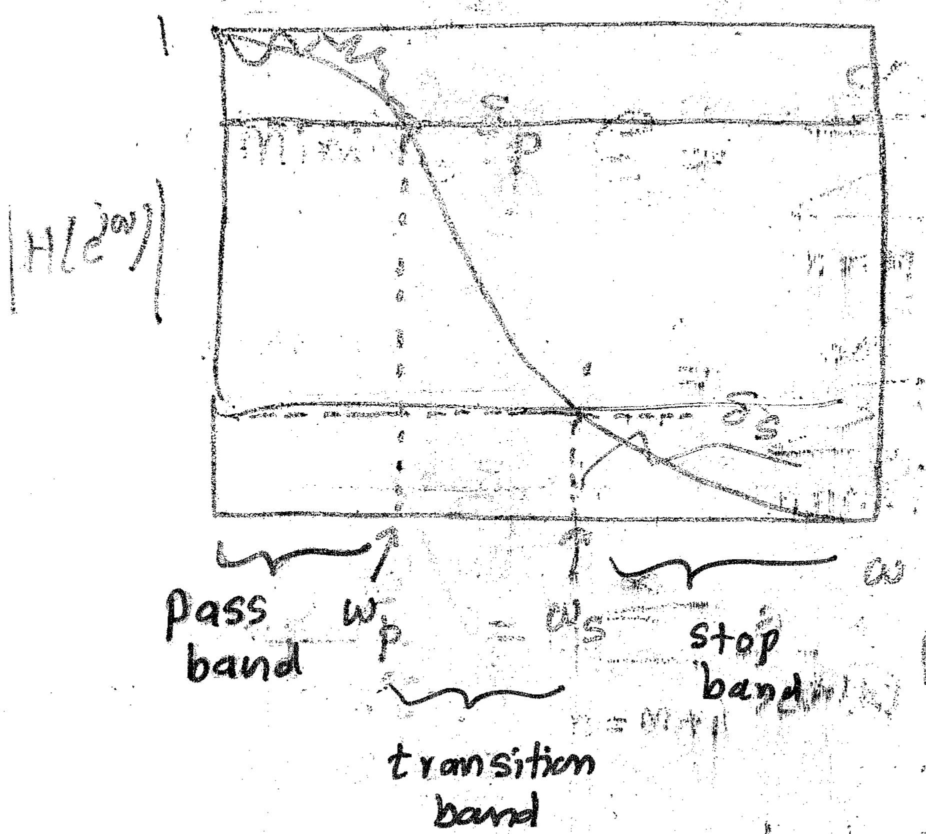
$\Rightarrow N = 5.88 \longrightarrow$ Set $N = 6$.

(Filter order known).

<u>Using $N = 6$</u>

$\Longrightarrow \Omega_c = 176.2$ rad/s (28.0 Hz)

Filter Design specifications in the digital Domain



ω_p, ω_s : Edge Frequencies of the transition band

ω_p : Edge frequency of passband

ω_s : Edge frequency of stopband

δ_p : minimum passband gain

δ_s : maximum stopband gain

Analog Butterworth Filter ~~with~~, Order = 6.

$$H(s) = \frac{\Omega_c^6 (1)}{(s^2+0.5177\Omega_c s+\Omega_c^2)(s^2+1.4142\Omega_c s+\Omega_c^2)(s^2+1.9319\Omega_c s+\Omega_c^2)}$$

Method

Impulse Invariance Design:

* Substitute Ω_c in $H(s)$
* Get $H(s)$ in the partial fraction form
* $s_p \rightarrow e^{p_i T}$
* Recombine numerator / Denominator

* Bilinear Transform Technique

* Pre-Warping:

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$$

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$$

Determine N and Ω_c as before Ω_{cp}

Substitute $s = \frac{2}{T}\frac{z-1}{z+1}$ in $H(s)$,

$$e[n] = \begin{cases} x[n] + \sum_{\ell=1}^{N} a[\ell]\,x[n-\ell] - b[n] , & 0 \le n \le M \\ x[n] + \sum_{\ell=1}^{N} a[\ell] \cdot x[n-\ell] , & n \ge M+1 \end{cases}$$

Minimize: $\mathcal{E} = \sum_{n=M+1}^{\infty} |e[n]|^2$

$$= \sum_{n=M+1}^{\infty} e^*[n]\, e[n]$$

$$\frac{\partial \mathcal{E}}{\partial a^*[k]} = \sum_{n=M+1}^{\infty} \frac{\partial}{\partial a^*[k]} \left(e^*[n]\, e[n] \right) = 0$$

$$1 \le k \le N$$

<u>We know, for $n \ge M+1$</u>

$$e[n] = x[n] + \sum_{\ell=1}^{N} a[\ell]\, x[n-\ell]$$

$$e^*[n] = x^*[n] + \sum_{\ell=1}^{N} a^*[\ell]\, x^*[n-\ell]$$

$a^*[k]$ appears only once, when $\ell = k$

$$\sum_{n=M+1}^{\infty} \frac{\partial}{\partial a^*[k]} \left(e^*[n] e[n] \right) = \sum_{n=M+1}^{\infty} e[n] \cdot x^*[n-k]$$

Solving for b[k]

$$e[n] = x[n] + \sum_{\ell=1}^{N} a[\ell]\cdot x(n-\ell) - b[n]$$

$, \; 0 \le n \le M$

Set $e[k] = 0$, for $0 \le k \le M$

$$b[k] = x[k] + \sum_{\ell=1}^{N} a[\ell]\cdot x(k-\ell)$$

$\boxed{\text{Solve for } b[k]}$

* Less sensitive to noise than Padé

* M, N : Not known "a-priori"

* A method to design a filter

A method to model a signal

A method to estimate the power

Spectrum

FIR Filters

Sampling

$$h[n] = \begin{cases} h_n, & 0 \le n \le M-1 \\ \text{specified at} & \\ 0, & n > M-1 \end{cases}$$

Desired ... points

(Causal Filter)

$$H(z) = \sum_{k=0}^{M-1} h[k] z^{-k} \quad \text{FIR filter}$$

$$= \sum_{k=0}^{M-1} h_k z^{-k}$$

Response

$$= h[0] + h[1] z^{-1} + \cdots + h[M-1] z^{-(M-1)}$$

$$= z^{-(M-1)} \left(\underbrace{h[0] z^{M-1} + h[1] z^{M-2} + \cdots + h[M-1]}_{\text{Zeros } M-1} \right)$$

$M-1$ Poles at $z = 0$

$M = 3$

Symmetry

Assume M is odd

$$h[M-1-n] = h[n]$$

FIR Filter Design by Frequency Sampling

* Desired freq. response specified at M different points.

* Use the M-specified values to get the M-point (M-tap) FIR filter.

$h[n], \quad 0 \leq n \leq M-1$.

Desired Response

$$H_d(e^{j\omega}), \quad \omega = \omega_k = \frac{2\pi}{M} k$$

$$0 \leq k \leq \frac{M-1}{2} \quad (M\text{-odd})$$

$$0 \leq k \leq \frac{M}{2} - 1 \quad (M\text{-even})$$

$$H(e^{j\omega_k}) \triangleq H_d(e^{j\omega_k})$$

$$= \sum_{n=0}^{M-1} h[n] \cdot e^{-j\frac{2\pi}{M}k \cdot n}$$

$$h[m] = \frac{1}{M} \sum_{k=0}^{M-1} H_d(e^{j\omega_k}) e^{j\frac{2\pi k}{M} m}$$